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Non-perturbative methods

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Abstract

The current theoretical understanding of processes involving many weakly interacting bosons in the Standard Model and in model theories is discussed. In particular, such processes are associated with the baryon and lepton number violation in the Standard Model. The most interesting domain where the multiplicity of bosons is larger than the inverse of small coupling constant is beyond the scope of perturbation theory and requires a non-perturbative analysis.

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1 Introduction

The vastness of the field of non-perturbative methods in high-energy physics inevitably compels me to focus on a specific topic among those which attract a considerable interest and where a non-trivial development is likely in the near future. One such topic, which also has dominated the parallel session on non-perturbative methods, is related to multiboson phenomena in the electroweak physics and, more generally, in models with weak coupling. These phenomena are interesting because of two basic reasons. One is that it is with multiboson processes is associated the violation of the sum of the baryon (B) and the lepton (L) numbers in the Standard Model[1]. Therefore such processes determine the evolution of (B+L) at high temperature in the early universe[2]. Also as initially envisioned in early works[3, 4] and indicated by specific calculations[5, 6] the processes with (B+L) violation and production of many electroweak bosons might be in principle observable in high energy collisions in the multi-TeV energy range. The other, theoretical, reason is related to the old-standing problem of the factorial divergence of perturbation theory series, which dates back to the work of Dyson[7]. This problem looks to be a matter of a purely theoretical concern as long as the quantities under discussion are such that they appear at low orders, like the anomalous magnetic moment. For such quantities the inability in principle to find the exact result by the perturbative expansion, though disappointing, does not prevent from calculating in few first orders with an accuracy required by practical measurements or greater. However the problem of the inherent divergence of perturbative series becomes quite acute as soon as one considers processes at energies such that a large number of interacting particles can be produced, i.e. the processes which occur starting only from a high order of the perturbation theory, where the expansion becomes unreliable.

At present there is a general understanding that multiparticle electroweak processes with many bosons both in the initial and the final state (*many* \rightarrow *many* scattering), including those with (B+L) violation, are not suppressed at high temperature and thus they indeed determine the (B+L) history of the universe. On the other hand the understanding of the processes, in which many bosons are produced by two or few initial particles with high energy (*few* \rightarrow *many* scattering), is far from complete and there are arguments both pro and con the idea that at sufficiently high multiplicity of final particles such processes can have an observable cross section.

The contribution of the *many* \rightarrow *many* scattering at high temperature is described

within the WKB technique by expansion around non-trivial classical solutions of the field equations: sphalerons[8, 9] and, more generally, periodic instantons[10]. It is also believed that the *few* \rightarrow *many* scattering can be fully described by applying a WKB technique using special classical configurations of the field. However, this is still a conjecture, and a specific method of a full WKB analysis of the latter scattering has yet to be developed.

2 (B+L) violating electroweak processes

As is known[1], the electroweak interaction in the Standard Model does not conserve the sum of the baryon and the lepton numbers as a result of the triangle anomaly. The amount of the (B+L) violation in a process is determined by the change of the winding number N_{CS} of the electroweak gauge fields:

$$\Delta(B + L) = 6 \Delta N_{CS} \quad . \quad (1)$$

However, changing the winding number by one or several units requires presence (at least in the intermediate state) of the W and Z field configurations with energy of order m_W/α_W . This is usually illustrated by the sketch of the dependence of minimal energy of the field with a given N_{CS} as a function of N_{CS} shown in figure 1, and the field configuration, corresponding to the top of the barrier, is the so-called sphaleron[8] with energy E_{Sp} about 10 TeV [9].

If the energy E available in a process is much less than E_{Sp} then the only way in which (B+L) can be violated is due to quantum tunneling, which at $E = 0$ is described by the instanton[11] solution to the Euclidean field equations, whose action is $S_i = 2\pi/\alpha_W$. The amplitude of such process then contains the WKB tunneling factor $\exp(-2\pi/\alpha_W) \sim 10^{-80}$, which thus makes the process unobservable by any practical measure.

The plot in the figure 1 however invites the suggestion that once the available energy is close to or larger than E_{Sp} the suppression of (B+L) violating processes should weaken or disappear altogether. The two relevant situations where large energy is available in individual processes are high temperatures and high-energy particle collisions.

2.1 (B+L) violation at high temperature

As first realized by Kuzmin, Rubakov and Shaposhnikov[2] the rate with which the system traverses the sphaleron barrier in thermal equilibrium at a temperature $T < E_{Sp}$

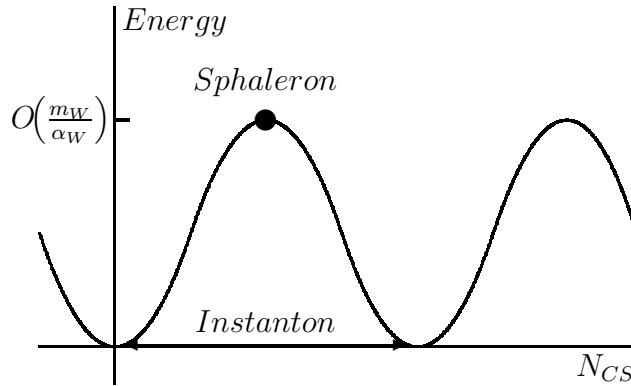


Figure 1: The dependence of minimal energy of the electroweak gauge field on its winding number.

is determined by the Boltzmann factor $\exp(-E_{\text{Sp}}(T)/T)$ and may become unsuppressed at temperatures larger than E_{Sp} . The dependence of E_{Sp} on the temperature arises through the temperature dependence of the vacuum expectation value of the Higgs field, which sets the electroweak energy scale. In particular the v.e.v. vanishes at the phase transition temperature T_c , above which the electroweak symmetry is restored. Thus at $T > T_c$ the sphaleron barrier is absent and the (B+L) violating processes may go without an exponential suppression. As a result [10, 12 - 14] the rate of change of the (B+L) density is given by

$$\Gamma_{\Delta(B+L)} = \begin{cases} C_1 \exp(-E_{\text{Sp}}(T)/T) & \text{if } T < T_c \\ C_2 \alpha_W^4 T & \text{if } T > T_c \end{cases} \quad (2)$$

where C_1 and C_2 are constants.

In particular, the prefactor C_1 for the ‘low’ temperature rate is determined by fluctuations of the fields near the sphaleron configuration. The initial calculations [15 - 17], which considered only the contribution of bosonic fluctuations, were most recently reanalyzed and extended[18] to include also the fermionic determinant. The contribution of the heavy top quark is found to significantly suppress the prefactor C_1 , which enables to somewhat relax the upper bound on the mass of Higgs boson in the minimal Standard Model, following from the requirement[10, 14, 19] that the (B+L) violating processes in the early universe immediately after the electroweak phase transition (i.e. just below T_c) do not wash out completely the baryon asymmetry, independently of the mechanism by

which it was created before or during the phase transition. Using their result for C_1 and $m_t = 174 \text{ GeV}$, Diakonov *et.al.* [18] find this upper bound to be $m_H < 66 \text{ GeV}$, which is only slightly higher than the lower bound $m_H > 58.4 \text{ GeV}$ [20] from a direct search at LEP. Therefore an improvement in the experimental search can either find the Higgs boson or close the gap of compatibility of the minimal model with the observed baryon asymmetry of the universe.

The shape of the sphaleron barrier in the presence of heavy top and the evolution of the energy levels of a heavy fermion were considered in detail in the contributed papers [21] and [22] respectively. Multi-sphaleron configurations are considered in [23] and electroweak strings, viewed as “stretched sphalerons” in [24]. However the role of the latter configurations in thermal equilibrium is yet to be clarified.

2.2 (B+L) violation in high-energy collisions

The sphaleron energy scale E_{Sp} is within (hopefully) reachable energies at prospective colliders. Therefore a most intriguing question arises as whether the exponential suppression of the (B+L) violating processes vanishes at an energy of order E_{Sp} in collision of two leptons or quarks. The difference between the high-temperature (B+L) violation and the processes induced by just two or few energetic particles is that in the former case the dominant contribution to the rate comes [13] from processes in the thermal bath, in which many soft particles with total energy $E \gtrsim E_{\text{Sp}}$ scatter into a final state of also soft particles with different (B+L), while in the latter case a coupling between the hard initial particles and soft modes of the field with a non-trivial topology is required.

Following the conjecture [13] that an enhancement of the cross section of (B+L) violating scattering may be associated with multiparticle final states, Ringwald [5] and Espinosa [6] pursued a calculation of a generic instanton-induced process of the type

$$\bar{f} + \bar{f} \rightarrow 10 f + n_W W + n_H H , \quad (3)$$

where n_W (n_H) is the multiplicity of produced gauge (Higgs) bosons and f stands for a quark or a lepton. (The presence in the instanton-induced scattering of twelve fermions: nine quarks and three leptons, one from each electroweak doublet is mandated by the anomaly condition in eq.(1), i.e. by the number of fermionic zero modes of an instanton.) The amplitude of the scattering (3) was found to factorially depend on the multiplicity of bosons: $A \sim n_W! n_H! \exp(-S_i)$, which lead to the argument [25] that the factorial enhancement may beat the exponential suppression at $n_{W,H} > O(1/\alpha_W)$ i.e. at energy

larger than $O(E_{Sp})$. In fact the growth with energy of the total cross section for (B+L) violating processes, observed in the early calculations[5, 6, 25], suggested[25] that this cross section may become strong: reach its unitarity limit at energies in the multi-TeV range.

2.2.1 “Holy grail” function.

By quite general scaling arguments[26, 27] the total cross-section of instanton-induced scattering should obey the scaling behavior

$$\sigma_{\Delta(B+L)}^{tot} \sim \exp \left[-\frac{4\pi}{\alpha_W} F \left(\frac{E}{E_0} \right) \right] , \quad (4)$$

where $E_0 \sim E_{Sp} \sim m_W/\alpha_W$. The function $F(\epsilon)$ is often termed as “holy grail” function. At $\epsilon = 0$ one has $F(0) = 1$, while the initial enhancement[5, 6] of the cross section due to opening multi-boson channels corresponds[28, 29] to the first non-trivial term in the expansion in ϵ : $F(\epsilon) = 1 - \frac{9}{8}\epsilon^{4/3} + \dots$, where $\epsilon = E/E_0$ with $E_0 = \sqrt{6}\pi m_W/\alpha_W \approx 18 \text{ TeV}$. The expansion in fact goes in powers of $\epsilon^{2/3}$, and by now two next terms are known[30 - 34]:

$$F(\epsilon) = 1 - \frac{9}{8}\epsilon^{4/3} + \frac{9}{16}\epsilon^2 + \frac{3}{32} \left(4 - 3 \frac{m_H^2}{m_W^2} \right) \epsilon^{8/3} \ln \epsilon + \dots \quad (5)$$

The latter two terms are determined by interaction between soft final particles. Starting from the term of order $\epsilon^{10/3}$ the “holy grail” function is also contributed by interactions between hard initial and soft final particles and by interaction between the initial hard particles[35 - 37].

Unfortunately, any finite number of terms in the expansion of $F(\epsilon)$ does not allow to assert the behavior of the function at finite $\epsilon \sim O(1)$. Therefore it is not known yet, whether the function $F(\epsilon)$:

- i) goes to zero at finite ϵ (finite energy),
- ii) goes to zero as $\epsilon \rightarrow \infty$, or
- iii) is bounded from below by a positive value.

Certainly, the most interesting phenomenologically is the first possibility, since then the cross section with (B+L) violation and multiboson production becomes observably large at a finite energy, while the most discouraging would be the last case, since then the cross section would stay exponentially suppressed at all energies.

The possibility *iii* was advocated[38 - 40] in terms of the so-called “premature unitarization”[40]. The argument is based on considering the interplay in the s -channel

$$\sigma^{tot} = \text{Im} \left[e^{-S_i} B(E) e^{-S_i} \right. \\ \left. e^{-S_i} B(E) e^{-S_i} B(E) e^{-S_i} B(E) e^{-S_i} \right] + \dots$$

Figure 2: Unitarization in the s -channel according to the model[40] of “premature unitarization”.

unitarity of the processes $few \rightarrow many$ and $many \rightarrow many$. The former processes are argued to be still weak (exponentially suppressed) when the processes $many \rightarrow many$ are at the unitarity limit, which effectively shuts off the further growth of the $few \rightarrow many$ cross section.

A somewhat simplified picture of this behavior is shown in figure 2, where the total cross section is represented as imaginary part of a $2 \rightarrow 2$ forward scattering amplitude through an instanton (I) - antiinstanton (\bar{I}) configuration. According to the model of “premature unitarization” [40] the total amplitude is given by summation over instanton - antiinstanton chains iterated in the s -channel, i.e. where all the total energy flows through the additional (anti)instantons. Each additional $I - \bar{I}$ pair brings in the factor $e^{-2S_i} B(E)$, where the “bond function” $B(E)$ is the multi-boson enhancement of the one-instanton-induced cross section observed in [5, 6, 25]. The summation over the $I - \bar{I}$ chains (figure 2) gives

$$\sigma^{tot} \sim e^{-S_i} \text{Im} \left[\frac{B(E)e^{-S_i}}{1 + \eta(B(E)e^{-S_i})^2} \right], \quad (6)$$

where $\eta = O(1)$ is a rescattering factor. If given by eq.(6), the cross section reaches its maximum when $B(E)e^{-S_i} = O(1)$ and its value at the maximum is of order e^{-S_i} , which corresponds to the lower bound of $1/2$ for the function $F(\epsilon)$. The presented reasoning is however oversimplified: it assumes that all the (anti)instantons in the chains have same fixed size. Relaxing this assumption leads[40] to a lower bound for $F(\epsilon)$, which is generally different from $1/2$.

At still higher energies the formula (6) gives a falling cross section. However this regime is unphysical: initial particle can shake off energy by emitting one or few hard

bosons, so that the energy in the collision gets back to the one corresponding to the maximum. (Emission of hard bosons suppresses the cross section by a few powers of the coupling constant, while the gain in the non-perturbative amplitude is exponential.) If indeed the “holy grail” function has a minimum at some energy, this would imply that above that energy the process can not be described by semiclassical methods, since emission of hard quanta becomes essential.

It turns out however, that the “premature unitarization” and thus the simple picture of the s -channel iteration of instanton-antiinstanton correlations is not mandatory and apparently depends on specifics of the theory. The known examples of simplified models, where the “holy grail” function is indeed bounded from below by $1/2$ are the Quantum Mechanical problem with a double well potential[41, 42] and the soft contribution to the scattering through a bounce[43, 44] in a $(1+1)$ dimensional model of one real field with metastable vacuum[45]. (It has been pointed out[46, 47] that in the latter model there is also a hard contribution to the bounce-induced scattering, for which the “holy grail” function goes to zero at the analog of the sphaleron energy.) Another example, where the “holy grail” function is bounded by a value, smaller than $1/2$, namely 0.160 , is the problem of catalysis of false vacuum decay in $(3+1)$ dimensions by collision of two (or few) particles[48]. In this problem the semiclassical probability reaches maximum at the top of the energy barrier.

2.2.2 Rubakov - Tinyakov approach.

The main difficulty in developing a semiclassical approach to the $few \rightarrow many$ scattering is the presence of hard quanta in the initial state, which state is thus not a semiclassical one. It has been suggested[49, 50] to circumvent this difficulty by considering a scattering, where a finite small number of particles in the initial state is replaced by $n_{i(initial)} = \nu/g^2$, where g is the coupling constant in the theory and ν is a parameter. For a finite ν the initial state of this kind can be treated semiclassically, and in the end the limit of the probability at $\nu \rightarrow 0$, or $\nu \rightarrow \text{const}/g^2$ is to be considered in order to relate to the process $few \rightarrow many$. Within such setting the “holy grail” function depends on ν : $F(\epsilon, \nu)$ and it is conjectured that its limit at $\nu \rightarrow 0$ is smooth, which conjecture is supported by high-order perturbative calculations around the instanton[51]. The central point of this approach is that the function $F(\epsilon, \nu)$ is determined from a solution to a well-defined boundary value problem[52] for classical field equations, although in essentially complex time.

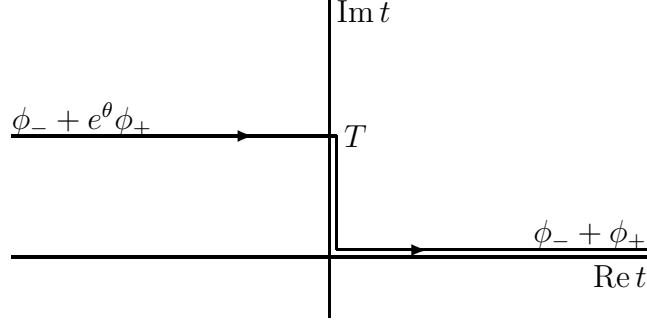


Figure 3: Contour in the complex time plane and boundary conditions for a classical solution, describing the scattering of a semiclassical initial state into multiparticle final states[52].

The classical solution that describes the path of largest probability in a model with one real field ϕ is evolving along the contour in the complex time plane shown in figure 3. At $\text{Re } t \rightarrow +\infty$ the solution is required to be real, thus its momentum components should be of the form $\phi(\mathbf{k}) = b_{\mathbf{k}} e^{-i\omega_k t} + b_{-\mathbf{k}}^* e^{i\omega_k t}$, while at $\text{Re } t \rightarrow -\infty$ the positive frequency part is rescaled by the parameter e^θ : $\phi(\mathbf{k}) = f_{\mathbf{k}} e^{-i\omega_k t} + e^\theta f_{-\mathbf{k}}^* e^{i\omega_k t}$. The parameter θ in the boundary condition and the parameter T of the contour (cf. figure 3) are Legendre-conjugate of respectively the multiplicity n_i and the total energy E of initial particles. Namely, if iS is the classical action on the whole contour (thus S is defined in the way that it is real in the Euclidean space), one finds[52]

$$n_i = 2 \frac{\partial S}{\partial \theta} , \quad E = 2 \frac{\partial S}{\partial T} . \quad (7)$$

Furthermore the “holy grail” function, entering the WKB estimate of the total cross section as $\sigma^{tot} \sim \exp(-g^{-2} F(\epsilon, \nu))$ is given by the Legendre transform of the action:

$$\frac{1}{g^2} F(\epsilon, \nu) = 2S - ET - n_i \theta . \quad (8)$$

Quite naturally, the formulated classical boundary value problem is not easily solvable, and a sufficiently good approximation to the solution is known only in a few models[53 - 55]. In particular the model, considered in [54], describes one scalar field in (1+1) dimensions with the potential

$$V(\phi) = \frac{m^2}{2} \phi^2 - \frac{m^2 v^2}{2} \exp \left[2\lambda \left(\frac{\phi}{v} - 1 \right) \right] , \quad (9)$$

Figure 4: The “holy grail” function normalized to $F(0, \nu) = 1$ in the (1+1) dimensional model[54] with exponential interaction.

where v and λ are dimensionless constants, which both are assumed to be large. The parameter $1/v$ is the small coupling constant of the perturbation theory in the vacuum $\phi = 0$. The negative sign of the interaction term implies that the energy is unbounded from below at large ϕ , thus the vacuum $\phi = 0$ is metastable, and is separated from the decreasing part of the potential by a barrier located at $\phi \approx v$, provided that $\lambda \gg 1$. Beyond the maximum the potential rapidly goes down, so that the potential essentially is a quadratic well with a “cliff”[54]. The metastability of the perturbative vacuum at $\phi = 0$ does not show up in calculations of the scattering amplitudes to any finite order of the perturbation theory, and it only arises through a non-perturbative effect: unitary “shadow” from the false vacuum decay, which makes this contribution analogous to instanton-induced scattering amplitudes in a Yang-Mills theory[45, 56]. The analog of the sphaleron energy is the height of the barrier separating two phases: $E_{\text{Sp}} = \text{const} \cdot mv^2$.

At large λ the potential (9) contains a sharp matching of the quadratic part (free field) and a steep exponential “cliff”, which enables[54] to solve the boundary value problem in the leading order in $1/\lambda$ and also to clarify the contribution of multi-instanton (multi-bounce) configurations. It has been found[54] that the multi-instanton configurations in this model are still not important when the one-instanton contribution becomes large. As a result the “holy grail” function, as shown in figure 4, reaches zero at finite energy, which energy increases when the semiclassical parameter of the initial state multiplicity $\nu = n_i v^2$ decreases. In figure 4 is also shown the behavior corresponding to the periodic instanton, which maximizes over n_i the rate of tunneling through the barrier in the processes $n_i \rightarrow n_f$ at given energy E [10].

2.2.3 Prospects for QCD hard processes.

It has been argued[57 - 60] that a manifestation of instanton-induced scattering in a weak coupling regime can be observed in hard processes in QCD. The suggestion is to search for final states in hadron collisions, which contain a large number of minijets, each with a typical invariant mass μ , such that $\alpha_s(\mu)$ is sufficiently small, e.g. $\mu \approx 4 \text{ GeV}$, so that $\alpha_s(\mu) \approx 0.25$. An instanton-induced process should involve production of typically $n_j \approx 4\pi/\alpha_s(\mu) \approx 50$ such jets, which requires energy in a parton - parton collision of somewhat higher than $n_j \mu \approx 200 \text{ GeV}$. The prospects of observing the instanton induced hard processes in QCD are certainly more phenomenologically attractive, since, unlike the electroweak case, the energy range is hopefully within the reach of LHC and also the cross section can be of a more encouraging magnitude, even if it is suppressed by an exponential factor, like $\exp(-2\pi/\alpha_s(\mu)) \sim 10^{-11}$ as suggested by the “premature unitarization” models. However the reality of observing these possible non-perturbative hard processes in QCD is still under discussion.

3 Multi-particle production in topologically trivial sector

The growth of the rate of the instanton-induced processes is associated with production of multiboson final states until at high multiplicity $n_f \sim 1/g^2$ the final state becomes not tractable perturbatively. A similar problem in fact arises[61, 62] at those high multiplicities in processes, which do not require contribution of field configurations with non-trivial topology, and thus are allowed in perturbation theory. This is related to the well known factorial growth of coefficients in the perturbation theory series[7]. Namely in the perturbation theory the total cross section for production of n bosons interacting with a weak coupling g is given, modulo the phase space suppression at finite energy, by

$$\sigma_n \sim n! (g^2)^n . \quad (10)$$

At small n , naturally, the cross section is decreasing with multiplicity. However at $n \sim 1/g^2$ the growth of $n!$ becomes faster than the decrease of $(g^2)^n$ and the behavior (10) would imply that the cross section starts to grow with multiplicity. Therefore the question: “If there is enough energy to produce $\gg 1/\alpha_W$ W , Z , H bosons, will they be actually produced with non-negligible cross section?” does not seem to be entirely

paradoxical or idle in view of eq.(10). The difficulty of answering this question is in that the lowest order equation (10) becomes inapplicable already at $n \sim 1/g$, since the loop corrections to σ_n are governed by the parameter $n^2 g^2$. The latter can be seen from the number of rescatterings between the final particles: $O(n^2)$, each having strength g^2 . In what follows we will discuss several steps that have been attempted toward answering the above question. It also may well be that a solution of the multiboson problem without the topological complications will provide an insight into the problem of (B+L) violation in high-energy collisions.

3.1 Multiboson amplitudes in $\lambda \phi^4$ theories.

One of simplest models, where the development of non-perturbative dynamics in multiboson amplitudes can be studied, is that describing one real scalar field with the $\lambda \phi^4$ interaction, whose potential is given by

$$V(\phi) = \frac{m^2}{2} \phi^2 + \frac{\lambda}{4} \phi^4. \quad (11)$$

If m^2 is positive the field has one vacuum state at $\langle 0|\phi|0\rangle = 0$ and the symmetry under sign reflection: $\phi \rightarrow -\phi$ is unbroken, while at negative m^2 there are two degenerate vacua $\langle 0|\phi|0\rangle = \pm v$ with $v = |m|/\sqrt{\lambda}$, which situation describes the spontaneous symmetry breaking (SSB).

3.1.1 Multiboson amplitudes at zero energy and momentum.

The simplest problem concerning multiboson amplitudes is, perhaps, that of calculating connected n -boson off-shell scattering amplitudes A_n , in which all the external particles have zero energy and momentum[63, 64]. The amplitude A_n can be written in terms of the connected part of the Euclidean-space correlator:

$$\frac{\int \left(\int \sigma(x) d^d x \right)^n \exp(-S[\phi]) \mathcal{D}\phi}{\int \exp(-S[\phi]) \mathcal{D}\phi}, \quad (12)$$

where $\sigma(x)$ is the deviation of the field ϕ from its vacuum mean value $\sigma(x) = \phi(x) - \langle 0|\phi|0\rangle$, and the integral $\int \sigma(x) d^d x$ is understood as the $p \rightarrow 0$ limit of the Fourier transform $\int \sigma(x) e^{ipx} d^d x$. Furthermore the connected part of the correlator (12) is conveniently given by the n -th logarithmic derivative of the generating functional $Z(j) =$

$\int \exp(-S[\phi] + \int j \sigma(x) d^d x) \mathcal{D}\phi$ with a constant source j :

$$A_n = \left(\frac{d}{dj} \right)^n \ln Z(j) \Big|_{j=0}. \quad (13)$$

Introduction of a constant source is equivalent to replacing the original potential $V(\phi)$ by $V(\phi) - j \sigma$. Furthermore, for a constant source $\ln Z(j)$ is related to the energy of the vacuum $E(j)$ in the presence of j : $\ln Z(j) = -VT E(j)$, where VT is the normalization space-time volume. Thus according to eq.(13) the asymptotic at large n behavior of the amplitudes A_n is related to the position j_c of the nearest to $j = 0$ singularity of $E(j)$ in the complex j plane: $A_n \sim n! j_c^{-n}$. At the classical level the position of the singularity is determined by the value of j , at which two solutions of the equilibrium equation $dV/d\phi = j$ coincide. For the potential (11) this happens at $j_c = \pm i \sqrt{4/27\lambda} m^3$, which determines the asymptotic behavior[86, 64] of the tree-level amplitudes A_n :

$$|A_n^{\text{tree}}| \sim n! \left(\frac{27}{4} \frac{\lambda}{|m|^6} \right)^{n/2}. \quad (14)$$

In a theory with unbroken symmetry the quantum loops modify $E(j)$ according to the Coleman-Weinberg potential[65] thus shifting and modifying the singularity in the j plane. However these corrections neither eliminate the singularity nor bring it to $j = 0$. The shift of the position can be absorbed in normalization of λ and m , while the modification of the type of the singularity only affects sub-leading in n factors, so that the leading behavior in eq.(14) is not modified by quantum corrections in a theory with unbroken symmetry.

The situation with quantum effects in a theory with SSB is drastically different: non-perturbatively the point $j = 0$ is in fact a branch point of the vacuum energy $E(j)$ for either of the vacua. Indeed, if, for definiteness, one choses to consider the amplitudes A_n in the ‘left’ vacuum: $\langle 0|\phi|0\rangle = -v$ with $v = |m|/\sqrt{\lambda}$, and follows the dependence of its energy on j , one finds that this state is stable for real $j < 0$ and is metastable at arbitrarily small positive j . Thus at $j > 0$ the energy $E(j)$ acquires an imaginary part given by the decay rate of the metastable vacuum. In this situation the Taylor expansion of $E(j)$ is asymptotic and the coefficients are determined by the decay rate in the presence of an infinitesimal positive source term. In this situation the calculation[66] of the false vacuum decay rate in the thin wall approximation is applicable exactly. Thus one can readily find the exact non-perturbative asymptotic behavior of the amplitudes

A_n at large n in a theory in d space-time dimensions[64]:

$$A_n \sim \left(\frac{n d}{d-1} \right)! \left(\frac{c_d \lambda}{|m|^6} \right)^{\frac{n}{2}} \left(\frac{\lambda}{|m|^{4-d}} \right)^{\frac{n}{d-1}} \quad (15)$$

with

$$c_d = \left[\frac{3^d}{2^{2d-1}} \frac{\Gamma(d/2)}{d^{d-1} \pi^{d/2}} \right]^{\frac{2}{d-1}}.$$

The factorial behavior in eq.(15), if interpreted in terms of loop graphs in perturbation theory, corresponds to contribution of graphs with $n/(d-1)$ loops.

The considered off-shell amplitudes A_n are not physical. However one can draw from the described exercise at least two, possibly important, theoretical conclusions about multiboson amplitudes:

- the $n!$ behavior suggested by the tree-level analysis is not necessarily eliminated and may even be enhanced in the exact result, and
- the large n behavior of multiboson amplitudes does not have to be universal and may in fact be very sensitive to details of the theory.

3.1.2 Production of on-shell multiparticle states at and above threshold. Tree graphs.

More physical, than the previously discussed off-shell amplitudes, are the amplitudes of processes, where n on-shell bosons are produced by a highly virtual field ϕ ($1 \rightarrow n$ process): $a_n = \langle n | \phi(0) | 0 \rangle$. (These e.g. can be related to the reaction $e^+ e^- \rightarrow n H$.) As will be explained, it turns out that one can explicitly find the sum of all tree graphs and all one-loop graphs for these amplitudes at any n , provided that the final bosons are exactly at rest in the c.m. system. Also the summation of two- and higher- loop graphs is in principle possible for this kinematical arrangement, however a calculation with a finite number of loops is inevitably plagued by the breakdown of the perturbation theory at large n . Thus far three methods have been used in calculation of the threshold amplitudes of the $1 \rightarrow n$ processes: the Landau WKB method, the recursion equations, and the functional technique.

Landau WKB method[67, 68] is used in Quantum Mechanics for calculating transition matrix elements between strongly different levels. (For a field theory derivation of this technique see [69].) In the tree graphs for the threshold $1 \rightarrow n$ amplitudes all the external and internal lines carry no spatial momentum. Thus the problem is reduced to

dynamics of only one mode of the field with spatial momentum $\mathbf{p} = 0$, i.e. to a Quantum Mechanical problem. This approach yields the result[70] for the sum of the tree graphs at the threshold with accuracy up to terms $O(1/n^2)$ at large n . (Application of the Landau WKB technique in the problem of multiboson amplitudes is also discussed in [71, 41, 72, 42].)

Recursion equations[64] for the amplitudes $a(n)$ arise from inspecting the construction of Feynman graphs. For the simplest case of tree graphs in $\lambda\phi^4$ theory the algebraic form of the equations is

$$(n^2 - 1) \frac{a(n)}{n!} = \lambda \sum_{\text{odd } n_1, n_2} \frac{a(n_1)}{n_1!} \frac{a(n_2)}{n_2!} \frac{a(n - n_1 - n_2)}{(n - n_1 - n_2)!}, \quad (16)$$

where the sum runs over odd n_1 and n_2 as well as n is odd, since due to the unbroken sign reflection symmetry the parity of the number of particles is conserved. Also the mass m in eq.(16) is set to one, since it can be restored in the final result from dimensional counting. The solution to the equations (16) reads as[64]

$$a(n) = n! (\lambda/8m^2)^{\frac{n-1}{2}}, \quad (17)$$

which can be found by applying the regular method of generating functions[73]. For the theory with SSB the recursion equations are modified by the presence of cubic vertices. The result for the amplitudes in the theory with SSB is[73]

$$a(n) = -n! (2v)^{1-n} \quad (18)$$

The recursion method can be extended to other theories[74] as well as to loop graphs[75, 76] and to an analysis of higher loops[77]. However a more convenient method for further analysis is the one suggested by Brown[78] and is based on a functional technique. Before proceeding to discussing this method and its further applications we report on estimates of the tree amplitudes above the threshold and thus of the total probability of the processes $1 \rightarrow n$ at a high energy E .

3.1.3 Lower bound for cross section at the tree level.

The tree graphs for the processes $1 \rightarrow n$ in a $\lambda\phi^4$ theory all have the same sign[61, 62]. The decrease of the amplitude above the threshold is thus determined by the increasing

virtuality of the propagators in those graphs, which depends on the kinematics of the final state. One can thus find a lower bound on the tree amplitudes above the threshold in a restricted part of the final-particle phase space[79 - 81], which gives a lower bound on the total probability of the process. In particular, if the kinematical restriction is chosen[79, 80] as the condition that the c.m. energy of each individual particle in the final state does not exceed ω , then in this region of the phase space the tree amplitude $A(1 \rightarrow n)$ is larger than the threshold amplitude $a(n)$ in which the physical mass M of the scalar boson is replaced by $\frac{9}{8}\omega$ in the theory without SSB (in which case $M = m$) and by $\frac{4}{3}\omega$ in the theory with SSB (where $M = \sqrt{2}|m|$). The cut off energy ω is then optimized for each value of the total energy E and multiplicity n in order to find the largest lower bound on the total probability

$$\sigma_n = \int |A(1 \rightarrow n)|^2 d\tau_n , \quad (19)$$

which is given by the integral over the n particle phase space τ_n . As a result the lower bound on σ_n is found[80] in the scaling form

$$\sigma_n > \exp \left[\frac{4\pi^2 c}{\lambda} f(\epsilon, \nu) \right] , \quad (20)$$

where $\nu = n M/E$ is the ratio of the multiplicity n to its maximal possible value E/M , $\epsilon = E/E_0$ with E_0 being an analog of the ‘sphaleron’ energy: $E_0 = 4\pi^2 c M/\lambda$, and the constant c in these formulas is $c = 9$ ($c = 8/3$) for a theory without (with) SSB. The calculated[80] behavior of the function $f(\epsilon, \nu)$ is shown in figure 5, which thus illustrates and quantifies the interplay between the $n!$ and the power of small coupling constant, discussed in connection with eq.(10). The function $f(\epsilon, \nu)$ displays a normal perturbative maximum at zero multiplicity. However, as energy grows, and production of high multiplicity states becomes unsuppressed kinematically, this function develops a second maximum, which at larger energies eventually crosses zero with an apparent violation of unitarity.

It is interesting to notice that the kinematical suppression of multiparticle final states is quite essential and shifts the energy, at which the tree graphs violate unitarity significantly higher than one would guess from a simple estimate $E_{crit} \approx 4\pi M/\lambda$. If applied to a multi-Higgs production in the Standard Model the lower bound (20) breaks unitarity at $E_{crit} \approx 15.5 (32\pi^2 M_H/\lambda) \approx 1000 \text{ GeV} (200\text{ GeV}/M_H)$.

It should be also mentioned that it is quite likely that the scaling behavior (20) at a given large multiplicity n also holds for the actual cross section, which point is recently

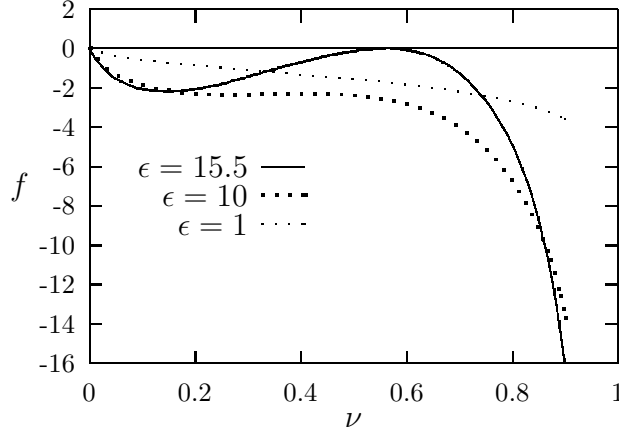


Figure 5: The function $f(\epsilon, \nu)$ vs. ν at several characteristic values of energy: low energy, no secondary maximum ($\epsilon = 1$), the secondary maximum just developed ($\epsilon = 10$), the secondary maximum becomes global and is just above the unitarity limit ($\epsilon = 15.5$)[80].

strongly emphasized in [82]. The function $f(\epsilon, \nu)$ can thus be called differential in n “holy grail” function.

3.2 Generating field technique.

3.2.1 Tree level.

A more convenient and more conceptually transparent technique for dealing with tree-level threshold multiboson amplitudes was suggested by Brown[78] and was later extended to calculation of one-loop[83, 84] and higher quantum effects[85, 86] in these amplitudes. The technique is based on the standard reduction formula representation of the amplitude through the response of the system to an external source $\rho(x)$, which enters the term $\rho\phi$ added to the Lagrangian.

$$\langle n|\phi(x)|0\rangle = \left[\prod_{a=1}^n \lim_{p_a^2 \rightarrow m^2} \int d^4x_a e^{ip_a x_a} (m^2 - p_a^2) \frac{\delta}{\delta \rho(x_a)} \right] \langle 0_{out}|\phi(x)|0_{in}\rangle^\rho|_{\rho=0} \quad , \quad (21)$$

the tree-level amplitude being generated by the response in the classical approximation, i.e. by the classical solution $\phi_0(x)$ of the field equations in the presence of the source.

For all the spatial momenta of the final particles equal to zero it is sufficient to consider the response to a spatially uniform time-dependent source $\rho(t) = \rho_0(\omega) e^{i\omega t}$ and take the on-mass-shell limit in eq.(21) by tending ω to m . The spatial integrals in eq.(21) then give the usual factors with the normalization spatial volume, which as usual is set to one, while the time dependence on one common frequency ω implies that the propagator factors and the functional derivatives enter in the combination

$$(m^2 - p_a^2) \frac{\delta}{\delta \rho(x_a)} \rightarrow (m^2 - \omega^2) \frac{\delta}{\delta \rho(t)} = \frac{\delta}{\delta z(t)} \quad , \quad (22)$$

where

$$z(t) = \frac{\rho_0(\omega) e^{i\omega t}}{m^2 - i\epsilon - \omega^2} \quad (23)$$

coincides with the response of the field to the external source in the limit of absence of the interaction, i.e. of $\lambda = 0$. For a finite amplitude ρ_0 of the source the response $z(t)$ is singular in the limit $\omega \rightarrow m$. The crucial observation of Brown[78] is that, since according to eq.(22) we need the dependence of the response of the interacting field ϕ only in terms of $z(t)$, one can take the limit $\rho_0(\omega) \rightarrow 0$ simultaneously with $\omega \rightarrow m$ in such a way that $z(t)$ is finite: $z(t) \rightarrow z_0 e^{imt}$.

Furthermore, to find the classical solution $\phi_0(x)$ in this limit one does not have to go through this limiting procedure, but rather consider directly the on-shell limit with vanishing source. The field equation with zero source in $\lambda\phi^4$ theory without SSB is

$$\partial^2 \phi + m^2 \phi + \lambda \phi^3 = 0 \quad . \quad (24)$$

For the purpose of calculating the matrix element in eq.(21) at the threshold one looks for a solution of this equation which depends only on time and contains only the positive frequency part with all harmonics being multiples of e^{imt} , which condition is equivalent to requiring that $\phi(t) \rightarrow 0$ as $\text{Im } t \rightarrow +\infty$. The solution satisfying these conditions reads as[78]

$$\phi_0(t) = \frac{z(t)}{1 - (\lambda/8m^2)z(t)^2} \quad (25)$$

According to equations (22) and (21) the n -th derivative of this solution with respect to z gives the matrix element $\langle n|\phi(0)|0\rangle_0$ at the threshold in the tree approximation:

$$\langle 2k+1|\phi(0)|0\rangle_0 = \left(\frac{\partial}{\partial z} \right)^{2k+1} \phi_0 \Big|_{z=0} \quad , \quad (26)$$

which reproduces the result in eq.(17).

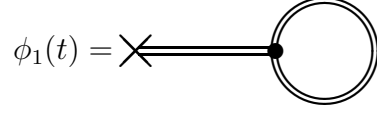


Figure 6: The tadpole graph in the background field ϕ_0 for calculating one-loop correction to ϕ_0 . The double lines and the heavy dot represent respectively the Green's function and the vertex in the background field.

For the case of theory with SSB the solution reads as

$$\phi_0(t) = -v \frac{1 + z/2v}{1 - z/2v} , \quad (27)$$

which reproduces the tree amplitudes in eq.(18). In this case $z(t) = e^{iMt}$, where $M = \sqrt{2}|m|$ is the mass of physical scalar boson.

3.2.2 One-loop level.

To advance the calculation to the one-loop level one has to calculate the first quantum correction $\phi_1(t)$ to the classical background field ϕ_0 . This amounts[83] to evaluating the tadpole graph of figure 6, where both the Green's function and the vertex are calculated in the external background field ϕ_0 . The green's function $G(x; x')$ satisfies the equation

$$\left(\partial^2 + m^2 + 3\lambda \phi_0(t)^2 \right) G(x; x') = -i \delta(x - x') , \quad (28)$$

in which the differential operator in the Minkowski time contains explicitly complex field ϕ_0 (cf. eq.(25) or eq.(27)). A straightforward rotation to the Euclidean time, $it \rightarrow \tau$, is problematic, since the background field then develops a pole at a real τ . The acceptable solution is achieved by simultaneously rotating and shifting the time axis in eq.(28) in such a way that $-\lambda z(t)^2/8m^2 \rightarrow \exp(2m\tau)$ for the theory without SSB, and $-z(t)/2v \rightarrow \exp(M\tau)$ for the theory with SSB. In terms of thus defined τ the background field has the form $\phi_0(\tau) = i\sqrt{2/\lambda}m/\cosh(m\tau)$ (no SSB) and $\phi_0(\tau) = v \tanh(M\tau/2)$ (with SSB). In both cases the term $\phi_0(t)^2$ in equation (28) is real and non-singular. After applying the standard decomposition of the Green's function over the conserved in the background $\phi_0(t)$ spatial momentum \mathbf{k} :

$$G(\tau, \mathbf{x}; \tau', \mathbf{x}') = \int G_\omega(\tau, \tau') e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} \frac{d^{d-1}k}{(2\pi)^{d-1}} , \quad (29)$$

one arrives for the case of no SSB at the well-known in Quantum Mechanics equation

$$\left(-\frac{d^2}{d\tau^2} + \omega^2 - \frac{6}{(\cosh \tau)^2}\right) = \delta(\tau - \tau') \quad (30)$$

with $\omega = \sqrt{\mathbf{k}^2 + 1}$ and the mass m set to one. (For the theory with SSB one gets the same equation with a rescaled ω [84].) Thus the problem of finding the first quantum correction $\phi_1(t)$ to the background field is completely solvable on the τ axis, and the solution can then be extended to the whole complex plane of t by analytical continuation. For the theory with no SSB the result[83] for the amplitudes $a(n)$ at the one-loop level reads as

$$a(n)_{0+1} = a(n)_0 \left(1 - (n-1)(n-3) \frac{3\lambda}{32} F\right), \quad (31)$$

where

$$F = \frac{\sqrt{3}}{2\pi^2} \left(\ln \frac{2 + \sqrt{3}}{2 - \sqrt{3}} - i\pi\right). \quad (32)$$

The analog of this result for the SSB case is[84]

$$a(n)_{0+1} = a(n)_0 \left(1 + n(n-1) \frac{\sqrt{3}\lambda}{8\pi}\right). \quad (33)$$

3.3 Nullification of threshold amplitudes.

The equations (31) and (33) display a remarkable feature: in spite of the presence of an intermediate state with two bosons in one-loop graphs, their contribution to the amplitudes in the case of SSB is real, while the factor F in eq.(32) is an easily recognizable threshold factor for the $2 \rightarrow 4$ process with no indication of presence of other thresholds. Using the unitarity relation for the imaginary part of the loop graphs, one immediately concludes that this can only be if the tree amplitudes of the on-shell processes $2 \rightarrow n$ are all zero at the threshold for $n > 4$ in the theory without SSB[83, 87] and for $n > 2$ in the theory with SSB[84].

This can be traced to the special properties of the reflectionless potential $-6/(\cosh \tau)^2$ in equation (30) and generalized[88, 76] to other theories, where the problem of the $2 \rightarrow n$ scattering is reduced to finding the Green's function in the reflectionless potential $-N(N+1)/(\cosh \tau)^2$ with integer N . The known additional cases are the following:

- Linear σ model ($N = 1$): the tree-level amplitudes of the scattering $\pi\pi \rightarrow n\sigma$ are all zero at the corresponding thresholds for $n > 1$.

- Fermions with Higgs-generated mass: if $2m_f/m_H = N$ (integer), then all tree-level amplitudes of $f\bar{f} \rightarrow nH$ are zero at threshold for all $n \geq N$.
- Vector bosons with Higgs-generated mass: if $4m_V^2/m_H^2 = N(N+1)$, then all tree-level amplitudes with transversal vector bosons of $V_T V_T \rightarrow nH$ are zero at threshold for all $n > N$. For longitudinal vectors the same amplitudes of $V_L V_L \rightarrow nH$ are zero for $n = N$ [89] and for $n > N+1$ [76].

All these cases, except the one with longitudinal vectors, stem[88] from the generic interaction of two fields ϕ and χ of the form $\frac{\xi}{2}\phi^2\chi^2$ and the self-interaction of the field ϕ as described by the potential (11). Then if the ratio of the coupling constants satisfies the relation $2\xi/\lambda = N(N+1)$ with N integer, the tree-level threshold amplitudes of the processes $2\chi \rightarrow n\phi$ are all zero for $n > N$ in a theory with SSB and for $n > 2N$ in a theory with unbroken symmetry. This behavior somewhat resembles the nullification of inelastic amplitudes in the Sine-Gordon theory, where it is a consequence of a symmetry and is a deep property of the theory. In the theories considered here this is a much weaker property, which holds only at threshold and, generally, only at the tree level[90]. However the nullification in this case can be a consequence of a hidden symmetry, which holds at the classical level and/or has a more restricted scope. Thus far such symmetry has been revealed[91] only for the case of $N = 1$, where it can be traced to the symmetry of a system of two anharmonic oscillators, described by the potential

$$V(x, y) = \frac{\omega_1^2}{2}x^2 + \frac{\omega_2^2}{2}y^2 + \frac{\lambda}{4}(x^2 + y^2)^2. \quad (34)$$

If the frequencies ω_1 and ω_2 were equal, the model would have an $O(2)$ symmetry, corresponding to conservation of the angular momentum $Q = \dot{x}y - x\dot{y}$. However even for $\omega_1 \neq \omega_2$ the symmetry persists[91] corresponding to conservation of the invariant $\frac{\lambda}{4}Q^2 + (\omega_1^2 - \omega_2^2)(\frac{1}{2}\dot{y}^2 + \frac{\omega_2^2}{2}y^2 + \frac{\lambda}{4}y^4 + \frac{\lambda}{4}x^2y^2)$.

It should be also noted that if the ratio $2\xi/\lambda$ does not satisfy the above mentioned condition, the threshold amplitudes of the processes $2\chi \rightarrow n\phi$ display[92] a ‘normal’ factorial growth with n .

3.4 Non-perturbative analysis.

The $n^2\lambda$ behavior of the loop corrections in the equations (31) and (33) convinces us that the perturbation theory is of little help in finding the amplitudes at large n and a true non-perturbative analysis is required. It turns out that to a certain extent such

analysis can be performed for the $\lambda\phi^4$ theory with SSB. In terms of the variable τ the problem of calculating the threshold amplitudes $a(n) = \langle n|\phi(0)|0\rangle$ reduces[85] to a well defined Euclidean-space problem of calculating the quantum average $\Phi(\tau)$ of the field

$$\Phi(\tau) = \frac{\int (\int \phi(\tau, \mathbf{x}) d\mathbf{x} / \int d\mathbf{x}) e^{-S[\phi]} \mathcal{D}\phi}{\int e^{-S[\phi]} \mathcal{D}\phi} \quad (35)$$

with the kink boundary conditions, i.e. $\phi \rightarrow \pm v$ as $\tau \rightarrow \pm\infty$. The average field then expands at $\tau \rightarrow -\infty$ in the series

$$\Phi(\tau) = \sum_{n=0}^{\infty} c_n e^{nM\tau} \quad (36)$$

and the threshold amplitudes are given by $a(n) = n!c_n/c_1^n$, where the coefficient c_1 describes the one-particle state normalization: $c_1 = \langle 1|\phi(0)|0\rangle$.

Due to the fact that the classical kink solution provides the absolute minimum for the action under specified boundary conditions, the path integrals in eq.(35) are well defined (no negative modes) and thus the average field $\Phi(\tau)$ is real in any finite order of the perturbation theory. Thus the coefficients c_n of the expansion (36) are real too. Therefore the amplitudes $a(n)$ are real to any finite order in λ . As we have seen at the one loop level this implies the nullification of the tree amplitudes of $2 \rightarrow n$ for $n > 2$. In higher loops this implies a relation between the amplitudes of the processes $k \rightarrow n$ with different k . The only exception is the particular case of $n = 3$, for which the imaginary part of the $a(3)$ can be contributed only by the two-boson intermediate state. Since the imaginary part is vanishing, one concludes[85] that the $2 \rightarrow 3$ amplitude is vanishing at the threshold in all orders in λ .

The function $\Phi(\tau)$ given by the expansion (36) is manifestly periodic: $\Phi(\tau+2i\pi/m) = \Phi(\tau)$. Using this property and the boundary conditions at $\tau \rightarrow \pm\infty$ one finds[85] that the exact function $\Phi(\tau)$ necessarily has a singularity at a finite τ . Thus the expansion (36) has a finite radius of convergence, and thus the exact threshold amplitudes $a(n)$ grow at least as fast as $n!$. In other words, the quantum effects do not eliminate the factorial growth of $a(n)$.

As is indicated by the $n^2\lambda$ parameter of the perturbation theory for the coefficients c_n , the saddle point (SP) for the action $S[\phi]$, given by the \mathbf{x} -independent ‘domain wall’ with the kink profile is not the correct SP for calculating the coefficients c_n at large n . It has been argued[86] that the correct SP configuration is given by a spatially inhomogeneous field configurations in which the ‘domain wall’ is deflected towards negative τ by a

maximal amount h_0 . Then at a large negative τ the coefficients c_n are given by

$$c_n \sim e^{nMh_0} e^{-\mu(A-A_0)} , \quad (37)$$

where A is the area of the domain wall with deflection, A_0 is the same for the undistorted flat wall, and $\mu = |m|^3/3\lambda$ is the surface tension of the wall.

Finding the extremum of the ‘effective action’ in the exponent in eq.(37) exactly corresponds to finding the equilibrium configuration of a $(d-1)$ dimensional membrane with a force equal to nM applied at the point of maximum deflection. In general this problem has no real solution (the film gets punched). However a solution exists, where a part of the trajectory of the domain wall resides in the Euclidean space, and a part is in the Minkowski space[86]. The Minkowski-space part of the trajectory corresponds to evolution of a bubble made of a domain wall and having energy $E = nM$. The amplitudes $a(n)$ are then found as a sum of resonant contributions of the quantized levels of the bubble:

$$a(n) \sim n! \frac{e^{iI(E)/2}}{1 - e^{iI(E)}} \exp \left[f(d) E (E/\mu)^{\frac{1}{d-2}} \right] , \quad (38)$$

where $f(d)$ is a *positive* coefficient depending on the space-time dimension, and $I(E) = \oint p dr$ is the action of the bubble over one period of oscillation. Clearly the amplitudes $a(n)$ in eq.(38) have poles at the values of E satisfying the Bohr-Sommerfeld condition $I(E) = 2\pi N$.

The result in eq.(38) can be interpreted as that the growth of $a(n)$ is due to a strong coupling of the bubble states to the multi-boson states with all particles being exactly at rest. However, it is known from the numerical studies of mid-70s[93 - 95] that the lifetime of the bubbles is of order one in units of their period. Thus one should conclude[86] that there arises a non-perturbative form factor, which cuts off the integral over the phase space of the final bosons and makes the total probability of a moderate value, inspite of the extremely large value of the coupling to exactly static bosons. The total probability of the process $1 \rightarrow n$ in this picture is given by the probability of creating a bubble with energy E by a virtual field ϕ . This can be estimated[86] by the Landau WKB method and is found to be exponentially small:

$$\sigma(1 \rightarrow B(E)) \sim \exp \left[-2f(d) E (E/\mu)^{\frac{1}{d-2}} \right] , \quad (39)$$

where $f(d)$ is the same as in eq.(38). Thus one concludes that in the theory with SSB the total cross section of non-perturbative multiparticle production is extremely likely to be

exponentially small at high energy. However because of the usage of special properties of the theory with SSB it is not clear, whether this conclusion can be generalized to other theories, in particular, to the multi-boson production in the Standard Model.

4 Conclusions. Problems

The problem of multi-particle processes in theories with weak interaction is one of most challenging in the quantum field theory. In solving this problem we are most likely to find new methods of non-perturbative analysis of the field dynamics. As it stands now, there are mostly problems facing us, some of which are:

- It is not clear, to what extent the exponential suppression of the (B+L) violation in particle collisions is lifted at high energy: by a factor $1/2$ in the exponent, by a different factor, or completely. All these types of behavior are observed in simplified models.
- The $n!$ behavior of the amplitudes for production of n bosons survives the quantum effects, at least in some models. However this does not necessarily imply a catastrophic growth of the cross section.
- Peculiar zeros are observed in threshold amplitudes of multi-boson production. However it is not clear, whether they signal some deep properties or this is a mere coincidence.
- The classical field configurations give rise to multiparticle amplitudes. However their rôle in high-energy collisions is yet to be understood.

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References

- [1] G. 't Hooft, Phys. Rev. Lett. **37** (1976) 8.
- [2] V.A. Kuzmin, V.A. Rubakov and M.E. Shaposhnikov, Phys. Lett. **B155** (1985) 36.
- [3] N.H. Christ, Phys. Rev. **D21** (1980) 1591.
- [4] H. Aoyama and H. Goldberg, Phys. Lett. **B188** (1987) 506.

- [5] A. Ringwald, Nucl. Phys. **B330** (1990) 1.
- [6] O. Espinosa, Nucl. Phys. **B343** (1990) 310.
- [7] F.J. Dyson, Phys. Rev. **85** (1952) 861.
- [8] N. Manton, Phys. Rev. **D28** (1983) 2019.
- [9] F.R. Klinkhamer and N.S. Manton, Phys. Rev. **D30** (1984) 2212.
- [10] S.Yu. Khlebnikov, V.A. Rubakov and P.G. Tinyakov, Nucl. Phys. **B367** (1991) 334.
- [11] A.A. Belavin, A.M. Polyakov, A.S. Shvarts and Yu.S. Tyupkin, Phys. Lett. **59B** (1975) 85.
- [12] P. Arnold and L. McLerran, Phys. Rev. **D36** (1987) 581.
- [13] P. Arnold and L. McLerran, Phys. Rev. **D37** (1988) 1020.
- [14] A.I. Bochkarev and M.E. Shaposhnikov, Mod. Phys. Lett. **A2** (1987) 417.
- [15] L. Carson and L. McLerran, Phys. Rev. **D41** (1990) 647.
- [16] L. Carson, X. Li, L. McLerran and R.-T. Wang, Phys. Rev. **D42** (1990) 2127.
- [17] J. Baacke and S. Junker, Phys. Rev. **D49** (1994) 2055.
- [18] D. Diakonov, M. Polyakov, P. Pobylitsa, P. Seiber,
J. Schaldach and K. Goeke, Contrib. paper gls0901; Ruhr U. Bochum preprint:
RUB-TPII-05/94.
- [19] M.E. Shaposhnikov, Nucl. Phys. **B287** (1987) 757; Nucl. Phys. **B299** (1988) 797
- [20] Particle Data Group *Review of Particle Properties*, Phys. Rev. **D50** (1994) 1173.
- [21] J. Kunz and G. Nolte, Contrib. paper gls0925.
- [22] Y. Brihaye and J. Kunz, Contrib. paper gls0817; Utrecht preprint: THU-94/03.
- [23] B. Kleihaus and J. Kunz, Contrib. paper gls0771.
- [24] T. Vachaspati, Contrib. paper gls0938; Tufts Univ. preprint: TUTP-94-11.
- [25] L. McLerran, A.I. Vainshtein and M.B. Voloshin, Phys. Rev. **D42** (1990) 171.
- [26] P.B. Arnold and M.P. Mattis, Phys. Rev. **D42** (1990) 1738.
- [27] S.Yu. Khlebnikov, V.A. Rubakov and P.G. Tinyakov Nucl. Phys. **B350** (1991) 441.
- [28] V.I. Zakharov, Nucl. Phys. **B371** (1992) 637.
- [29] M. Porrati, Nucl. Phys. **B347** (1990) 371.

- [30] V.V. Khoze and A. Ringwald, Nucl. Phys. **B355** (1991) 351.
- [31] P. Arnold and M. Mattis, Phys. Rev. **D44** (1991) 3650.
- [32] A.H. Mueller, Nucl. Phys. **B364** (1991) 109.
- [33] D.I. Diakonov and V. Petrov, Proc. of the **XXVI** LNPI Winter School. 1991, p.8 (Leningrad 1991).
- [34] I. Balitsky and A. Shaefer, Nucl. Phys. **404** (1993) 639.
- [35] A.H. Mueller, Nucl. Phys. **B348** (1991) 310.
- [36] M.B. Voloshin, Nucl. Phys. **B359** (1991) 301.
- [37] M.P. Mattis, L. McLerran and L.G. Yaffe, Phys. Rev. **D45** (1992) 4294.
- [38] V.I. Zakharov, Nucl. Phys. **B353** (1991) 683.
- [39] G. Veneziano, Mod. Phys. Lett. **A7** (1992) 1661.
- [40] M. Maggiore and M. Shifman, Nucl. Phys. **B365** (1991) 161; Nucl. Phys. **B371** (1992) 177.
- [41] J.M. Cornwall and G. Tiktopoulos, Ann. Phys. (NY) **228** (1993) 365.
- [42] D. Diakonov and V. Petrov, Phys. Rev. **D50** (1994) 266.
- [43] S. Coleman, Phys. Rev. **D15** (1977) 2929; Err.- Phys. Rev. **D16** (1977) 1248.
- [44] C. Callan and S. Coleman, Phys. Rev. **D16** (1977) 1762.
- [45] M.B. Voloshin, Nucl. Phys. **363** (1991) 425.
- [46] V.G. Kiselev, Phys. Rev. **D45** (1992) 2929.
- [47] V.A. Rubakov, D.T. Son and P.G. Tinyakov, Nucl. Phys. **B404** (1993) 65.
- [48] M.B. Voloshin, Phys. Rev. **D49** (1994) 2014.
- [49] V.A. Rubakov and P.G. Tinyakov, Phys. Lett. **B279** (1992) 165
- [50] P.G. Tinyakov, Phys. Lett. **B284** (1992) 410.
- [51] A.H. Mueller, Nucl. Phys. **B401** (1993) 93.
- [52] V.A. Rubakov, D.T. Son and P.G. Tinyakov, Phys. Lett. **B287** (1992) 342.
- [53] D.T. Son and P.G. Tinyakov, Nucl. Phys. **B415** (1994) 101.
- [54] D.T. Son and V.A. Rubakov, Nucl. Phys. **B422** (1994) 195.
- [55] P.G. Tinyakov, Contrib. paper gls0671.

- [56] S.D.H. Hsu, Phys. Lett. **B261** (1991) 81.
- [57] V.I. Zakharov, Nucl. Phys. **B377** (1992) 501.
- [58] I.I. Balitsky and V.M. Braun, Phys. Rev. **D47** (1993) 1879.
- [59] I.I. Balitsky and V.M. Braun, Phys. Lett. **B314** (1993) 237.
- [60] I.I. Balitsky, Penn State Univ. preprint: PSU/TH/146 (1994).
- [61] J.M. Cornwall, Phys. Lett. **B243** (1990) 271.
- [62] H. Goldberg, Phys. Lett. **B246** (1990) 445.
- [63] H. Goldberg and M.T. Vaughn, Phys. Rev. Lett. **66** (1991) 1267.
- [64] M.B. Voloshin, Nucl. Phys. **B383** (1992) 233.
- [65] S. Coleman and E. Weinberg, Phys. Rev. **D7** (1973) 1888.
- [66] M.B. Voloshin, I.Yu. Kobzarev and L.B. Okun, Sov. J. Nucl. Phys. **20** (1975) 644.
- [67] L.D. Landau, Phys. Zs. Sowiet. **1** (1932) 88.
- [68] L.D. Landau and E.M. Lifshits, *Quantum Mechanics, Non-Relativistic Theory, Third edition*, section 52 (Pergamon 1977).
- [69] S.V. Iordanskii and L.P. Pitaevskii, Sov. Phys. JEPT **49** (1979) 386.
- [70] M.B. Voloshin, Phys. Rev. **D43** (1991) 1726.
- [71] M. Porrati, Phys. Lett. **B271** (1991) 148.
- [72] S.Yu. Khlebnikov, Phys. Lett. **B282** (1992) 459.
- [73] E.N. Argyres, R.H.P. Kleiss and C.G. Papadopoulos, Nucl. Phys. **B391** (1993) 42.
- [74] E.N. Argyres, R.H.P. Kleiss and C.G. Papadopoulos, Phys. Lett. **B296** (1992) 139.
- [75] M.B. Voloshin, Phys. Rev. **D47** (1993) 1712.
- [76] E.N. Argyres, R.H.P. Kleiss and C.G. Papadopoulos, Phys. Lett. **B302** (1993) 70;
Phys. Lett. **B319** (1993) 544.
- [77] E.N. Argyres, R.H.P. Kleiss and C.G. Papadopoulos, Phys. Lett. **B308** (1993) 292.
- [78] L.S. Brown, Phys. Rev. **D46** (1992) 4125.
- [79] M.B. Voloshin, Univ. of Minnesota preprint: TPI-MINN-92-27-T.
- [80] M.B. Voloshin, Phys. Lett. **B293** (1992) 389.
- [81] E.N. Argyres, R.H.P. Kleiss and C.G. Papadopoulos, Nucl. Phys. **B391** (1993) 57.

- [82] M.V. Libanov, V.A. Rubakov, D.T. Son and S.V. Troitskii, Moscow INR preprint: INR-866-94.
- [83] M.B. Voloshin, Phys. Rev. **D47** (1993) 357.
- [84] B.H. Smith, Phys. Rev. **D47** (1993) 3518.
- [85] M.B. Voloshin, Phys. Rev. **D47** (1993) 3525.
- [86] A.S. Gorsky and M.B. Voloshin, Phys. Rev. **D48** (1993) 3843.
- [87] M.B. Voloshin, Phys. Rev. **D47** (1993) 1712.
- [88] M.B. Voloshin, Phys. Rev. **D47** (1993) 2573.
- [89] B.H. Smith, Phys. Rev. **D49** (1994) 1081.
- [90] B.H. Smith, Phys. Rev. **D47** (1993) 5531.
- [91] M.V. Libanov, V.A. Rubakov and S.V. Troitskii, Phys. Lett. **B318** (1993) 134.
- [92] L.S. Brown and Cheng-Xing Zhai, Phys. Rev. **D47** (1993) 5526.
- [93] N.A. Voronov and I.Yu. Kobzarev, JETP Lett. **24** (1976) 532.
- [94] I.L. Bogolyubskii and V.G. Makhn'kov, JETP Lett. **24** (1976) 12.
- [95] T.I. Belova, N.A. Voronov, I.Yu. Kobzarev and
N.B. Konyukhova, Sov. Phys. JEPT **46** (1977) 846.

Discussion

A. Kataev, CERN and INR, Moscow:

Do the theoretical results discussed in your talk have any interesting phenomenological implications? Is it possible to study these non-perturbative effects experimentally?

M. Voloshin:

Possible effects in hard processes in QCD with production of many minijets might be observable at LHC energies. An experimental study of possible non-perturbative processes in the electroweak interactions may require an energy of about 1000 TeV.

V. Kuvshinov, Minsk:

It seems we have here new mechanisms for multiparticle production. For example, it can give contributions to the intermittency phenomenon. Is multiplicity important for $B + L$ violation? Or is only $n!$ important?

M. Voloshin:

The multiplicity is important through the $n!$, or, possibly, a stronger factor.